2020 B
Week 3
(Jan 26)

Last time we discussed the theoretical aspects of integral. In particular, we considered

Property I ⓐ f, g integrable. Then $\alpha f+\beta g$ is also integrable, and

$$\iint_D (\alpha f+\beta g)\,dA = \alpha \iint_D f\,dA + \beta \iint_D g\,dA.$$

ⓑ $f \geq 0 \Rightarrow \iint_D f\,dA \geq 0$

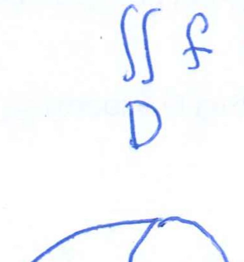
ⓐ can be generalized to $f_1, f_2, \cdots f_n$:

$$\iint_D (\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n)\,dA$$
$$= \alpha_1 \iint_D f_1\,dA + \alpha_2 \iint_D f_2\,dA + \cdots + \alpha_n \iint_D f_n\,dA.$$

Now,

Property II. $D = D_1 \cup D_2$, where the interiors of D_1, D_2 are empty. then

$$\iint_D f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA.$$



We show that this property follows from linearity.

Recall the formula (see assignment 2)

$$\chi_D = \chi_{D_1 \cup D_2}$$
$$= \chi_{D_1} + \chi_{D_2} - \chi_{D_1 \cap D_2}$$

Multiply it with f:

$$f\chi_D = f\chi_{D_1 \cup D_2} \qquad (1)$$
$$= f\chi_{D_1} + f\chi_{D_2} - f\chi_{D_1 \cap D_2}$$

We claim

$$\iint_D f\chi_D \, dA = \iint_D f \, dA, \qquad (2)$$

and

$$\iint_D f\chi_{D_1} \, dA = \iint_{D_1} f \, dA, \qquad (3)$$

$$\iint_D f\chi_{D_2} \, dA = \iint_{D_2} f \, dA, \qquad (4)$$

$$\iint_D f\chi_{D_1 \cap D_2} \, dA = 0 \quad , \qquad (5)$$

By linearity, we get from (1)

$$\iint_D f\chi_D = \iint_D f\chi_{D_1} + \iint_D f\chi_{D_1} - \iint_D f\chi_{D_1 \cap D_2}.$$

Using (2) – (5), it becomes

$$\iint_D f = \iint_{D_1} f + \iint_{D_2} f \quad \text{, done.}$$

Let R be a rectangle, $R \supset D$. Recall

$$\iint_D f\chi_D \, dA \overset{\text{def}}{=} \iint_R \widetilde{f\chi_D} \, dA$$ when $\widetilde{f\chi_D}$ is the

universal extension of $f\chi_D$ over D, ie,

$$\widetilde{f\chi_D}(x,y) = \begin{cases} f(x,y)\chi_D(x,y) & : (x,y) \in D \\ 0 & , (x,y) \notin D \end{cases}$$

but we have

$$= \begin{cases} f(x,y) & , (x,y) \in D \\ 0 & , (x,y) \notin D \end{cases}$$

$$= \text{the universal extension of } f \text{ over } D$$

$$\therefore \iint_D f\chi_D \, dA = \iint_R \widetilde{f\chi_D} \, dA$$

$$= \iint_R \tilde{f} \, dA$$

$$= \iint_D f \, dA$$, ie, (2) holds.

Next,

$$\iint_D f\chi_{D_1} \, dA \overset{\text{def}}{=} \iint_R \widetilde{f\chi_{D_1}} \, dA \text{, and}$$

$$\widetilde{f\chi_{D_1}}(x,y) = \begin{cases} f(x,y)\chi_{D_1}(x,y), & (x,y)\in D \\ 0, & (x,y)\notin D \end{cases}$$

$$= \begin{cases} f(x,y), & (x,y)\in D_1 \\ 0, & (x,y)\notin D_1 \end{cases}$$

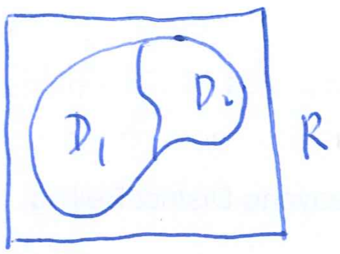
$=$ the universal extension of f over D_1

$$\therefore \iint_D f\chi_{D_1}\,dA = \iint_R \widetilde{f\chi_{D_1}}\,dA$$

$$= \iint_{D_1} f\,dA, \quad \text{ie, ③ holds.}$$

Similarly, ④ holds.

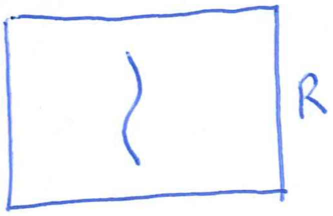
Finally, to prove ⑤ we consider a partition P on R.



Since $D_1\cap D_2$ is a curve, in each R_{jk} we can always pick a tag point $P_{jk}\notin D_1\cap D_2$

So $\chi_{D_1\cap D_2}(P_{jk})=0$, and $\widetilde{f\chi_{D_1\cap D_2}}(P_{jk})=0$ too.

then the Riemann sum



$$\sum_{j,k} \widetilde{f\chi_{D_1\cap D_2}}(P_{jk})\,\Delta x_j\,\Delta y_k = 0$$

$D_1\cap D_2$ is a curve

Let $\|P\|\to 0$,

$$\iint_D f\chi_{D_1\cap D_2}\,dA = \iint_R \widetilde{f\chi_{D_1\cap D_2}}\,dA$$

$$= 0, \text{ ie, ⑤ holds.}$$

- Area and Average of Functions

For a region $D \subset \mathbb{R}^2$, we define its area to be

$$|D| = \iint_D 1 \, dA .$$

Since for $f \geq 0$,

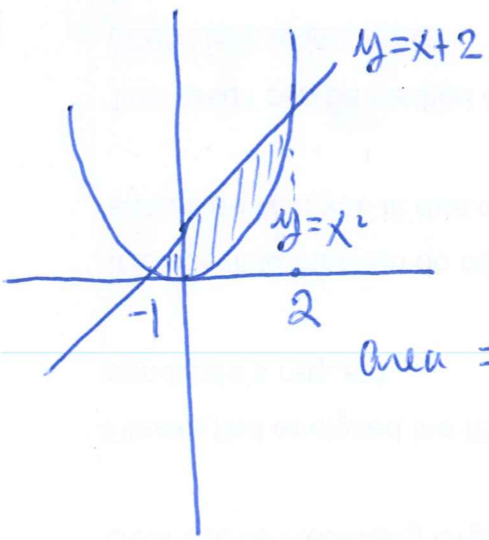
$\iint_D f \, dA$ is the volume bounded above by $z = f(x, y)$ over D,

when $f \equiv 1$, $\iint_D 1 \, dA = 1 \times$ the area of D. This justifies the above definition.

e.g. Find the area of the region D enclosed by $y = x^2$ and $y = x + 2$.

Sol. $\begin{cases} y = x^2 \\ y = x + 2 \end{cases}$

to get $x = -1$, or 2, so $y = 1, 4$, ie intersection pts are $(-1, 1)$ and $(2, 4)$



$$\text{area} = \int_{-1}^{2} \int_{x^2}^{x+2} 1 \, dy \, dx$$

$$= \int_{-1}^{2} (x + 2 - x^2) \, dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2}$$

$$= 9/2 \quad \#$$

e.g. Find the area of the playing field described by

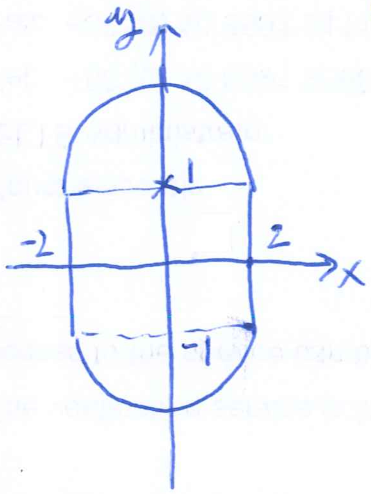
$$-1-\sqrt{4-x^2} \le y \le 1+\sqrt{4-x^2}$$
$$-2 \le x \le 2.$$

Here $g_2(x) = 1+\sqrt{4-x^2}$, $g_1(x) = -1-\sqrt{4-x^2}$

$y = 1+\sqrt{4-x^2}$ $\quad (y-1)^2 + x^2 = 4$,

So $g_2(x)$ is the upper semicircle (at $(0,1)$, radius 2)

Similarly, $g_1(x)$ is the lower semicircle (at $(0,-1)$, radius 2)



By symmetry,

$$\text{area} = 4\int_0^2 \int_0^{1+\sqrt{4-x^2}} dy\, dx$$
$$= 4\int_0^2 \left(1+\sqrt{4-x^2}\right) dx$$
$$\vdots$$
$$= 8 + 4\pi.$$

<u>Note</u>. The above approach is quite artificial. Indeed the simple geometry of this field yields directly the area.

- <u>polar coordinates</u>

Polar coordinates and the rectangular coordinates (Cartesian coordinates) are related by

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1} y/x , \quad \theta \in [0, 2\pi), \text{ or } (-\pi, \pi] \end{cases}$$

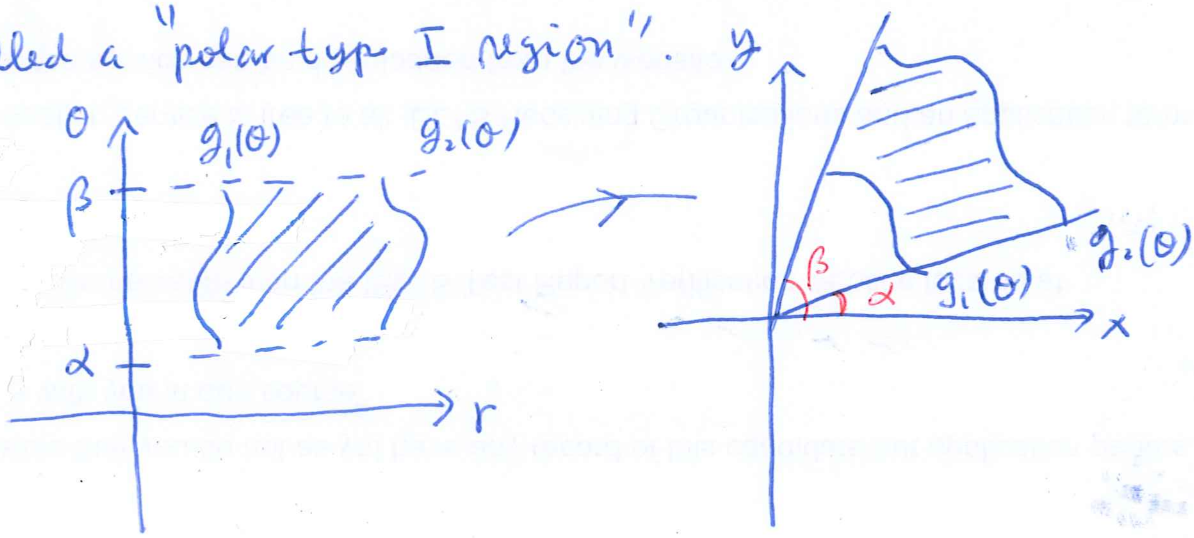
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

(For $(x,y) \neq (0,0)$, the representation is unique.)

A region described in the form

$$D = \left\{ (x,y) : \begin{array}{l} x = r\cos\theta,\ y = r\sin\theta, \\ g_1(\theta) \le r \le g_2(\theta),\ \alpha \le \theta \le \beta \end{array} \right\}$$

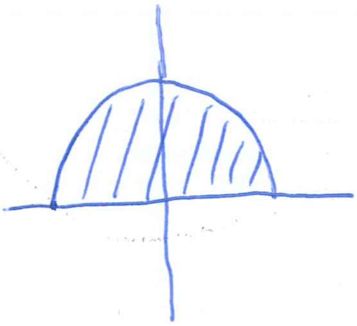
is called a "polar type I region".



(In (r,θ), a polar type I region is just a type II region.)

We show how to write regions as polar type I region.

e.g. Consider the region bdd by $x^2+y^2=1$, and the x-axis

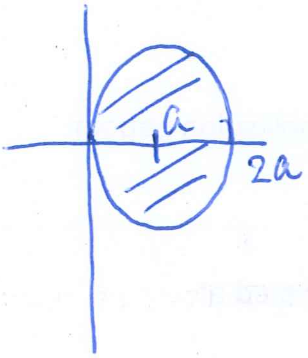


$D: \quad 0 \le \theta \le \pi$

$\quad\;\; 0 \le r \le 1$

e.g. the circle $(x-a)^2 + y^2 = a^2$.

Equation simplied to



$$r = 2a\cos\theta,$$

as $r \geq 0$, $\theta \in [-\pi/2, \pi/2]$ only.

$\therefore D$ is $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\qquad 0 \le r \le 2a\cos\theta$.

The formula is used for polar type I regions:

Let f be continuous / piecewise continuous in D.

$$\iint_D f \, dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta. \quad (\star)$$

e.g. Evaluate $\iint_D e^{x^2+y^2} dA$ where D is the region bdd by $x^2+y^2=1$ and the x-axis in the upper half plane.

We know from above that D:

$0 \le \theta \le \pi \quad (\alpha = 0, \beta = \pi)$

$0 \le r \le 1 \quad (g_1(\theta) = 0, g_2(\theta) = 1, \forall \theta)$

$$e^{(r\cos\theta)^2 + (r\sin\theta)^2} = e^{r^2}$$

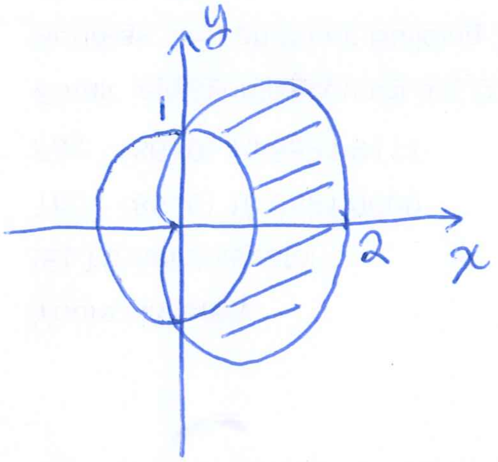
$$\therefore \iint_D e^{x^2+y^2} dA = \int_0^\pi \int_0^1 e^{r^2} r\, dr\, d\theta = \int_0^\pi \tfrac{1}{2} e^{r^2} \Big|_0^1 d\theta$$

$$\cdots = \frac{\pi}{2}(e-1).$$

Note. Use old way,

$$\iint_D e^{x^2+y^2}\,dA = \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx$$ hard to proceed further!

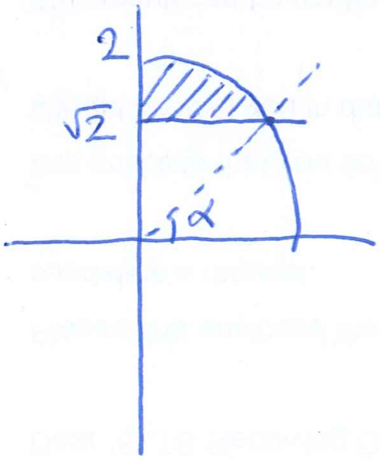
e.g. Express the region D which is lying inside the cardiod $r = 1+\cos\theta$ and outside the circle $r=1$ as a polar type I region.



Sketch the figures, we see that D:

$$-\pi/2 \le \theta \le \pi/2$$
$$1 \le r \le 1+\cos\theta.$$

e.g. Let D be the region bdd by $x^2+y^2=4$, $y=\sqrt{2}$, and the y-axis. Write it as a polar type I region.



The key is to determine α.

$$\begin{cases} x^2+y^2=4 \\ y=\sqrt{2} \end{cases} \Rightarrow \begin{array}{l} x=\sqrt{2} \\ y=\sqrt{2} \end{array}$$

$(\sqrt{2},\sqrt{2})$ is the intersection pt, so

$$\tan\alpha = \frac{\sqrt{2}}{\sqrt{2}} = 1, \text{ ie, } \alpha = \pi/4$$

Clearly, $\beta = \pi/2$

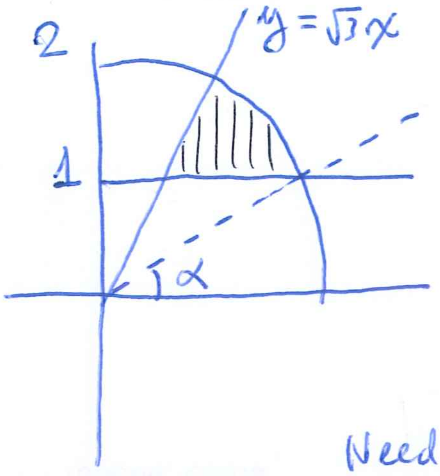
$$\therefore \pi/4 \le \theta \le \pi/2.$$

$y=\sqrt{2} \Rightarrow r = \sqrt{2}/\sin\theta$, $\therefore g_1(\theta) = \sqrt{2}/\sin\theta$.

Clearly, $g_2(\theta) = 2$.

$$\therefore \sqrt{2}/\sin\theta \le r \le 2.$$

e.g. Let D be the region bdd by $x^2+y^2=4$, $y=1$, and $y=\sqrt{3}x$. Find its area.



Find α. Solve

$$\begin{cases} x^2+y^2=4 \\ y=1 \end{cases}$$

to get intersection pt $(\sqrt{3},1)$

So $\tan\alpha = 1/\sqrt{3} \Rightarrow \alpha = \pi/6$

Need, solve $\begin{cases} x^2+y^2=4 \\ y=\sqrt{3}x \end{cases}$

to get intersection pt $(1,\sqrt{3})$

$$\tan\beta = \sqrt{3}/1 \Rightarrow \beta = \pi/3$$

$$\therefore \quad \pi/6 \le \theta \le \pi/3$$

$$g_1(\theta) = 1/\sin\theta, \quad g_2(\theta) = 2$$

$$\text{area of } D = \iint_D 1\, dA = \int_{\pi/6}^{\pi/3} \int_{1/\sin\theta}^{2} 1\, r\, dr\, d\theta$$

$$= \int_{\pi/6}^{\pi/3} \frac{1}{2}\left(4 - \frac{1}{\sin^2\theta}\right) d\theta$$

$$= \frac{1}{2}(4\theta + \cot\theta)\Big|_{\pi/6}^{\pi/3}$$

$$= \frac{\pi-\sqrt{3}}{3}. \#$$

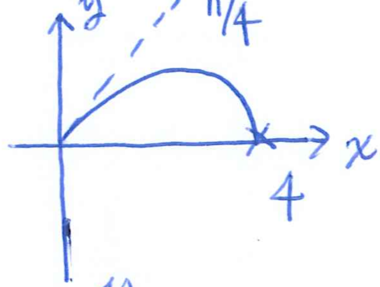
e.g. Find the area of the regions enclosed by the lemnisate (figure 8) $r^2 = 4\cos 2\theta$.

Focus on the range $[0, \pi/2]$

$\cos 2\theta \geq 0$ for $\theta \in [0, \pi/4]$

$\cos 2\theta < 0$ for $\theta \in (\pi/4, 0)$

θ increases from 0 to $\pi/4$, $\cos 2\theta$ decreases to 0

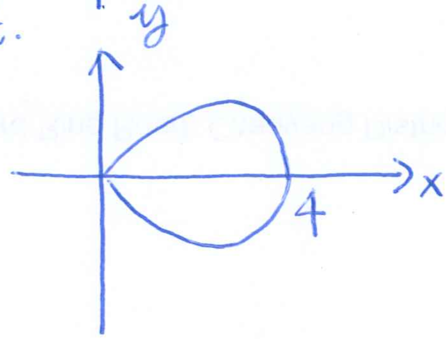


$\theta \mapsto -\theta$

$4\cos 2(-\theta) = 4\cos 2\theta$

as cosine is an even fcn

$\therefore$ the curve is symmetric w.r.t. the x-axis

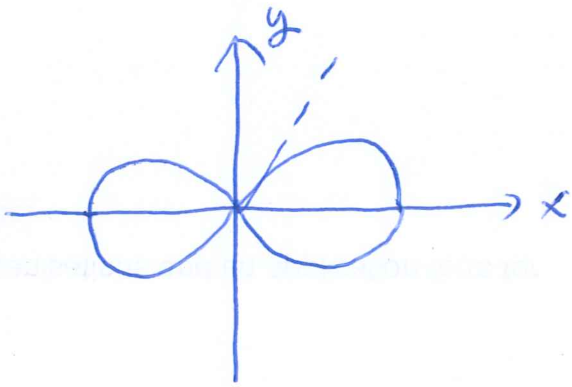


Next, $\theta \mapsto \theta + \pi$

$$4\cos 2(\theta+\pi) = 4\cos(2\theta + 2\pi)$$
$$= 4\cos 2\theta$$

as cosine is 2π-periodic.

So, the curve is itself after rotating π degree.



It consists of 2 regions.

$$\text{area} = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} 1 \, r \, dr \, d\theta$$

$$= 4\int_0^{\pi/4} \frac{1}{2} r^2 \Big|_0^{\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} \frac{1}{2} 4\cos 2\theta \, d\theta$$

$\vdots$

$$= 4 \ \#$$

e.g. Find the volume of solid bdd above by $z = 9 - x^2 - y^2$ over the disk $x^2 + y^2 \le 1$.

This disk is described by

$$0 \le \theta \le 2\pi$$
$$0 \le r \le 1$$

volume

$$\therefore \iint_D (9 - x^2 - y^2)\, dA = \int_0^{2\pi} \int_0^1 (9 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{9}{2} r^2 - \frac{r^4}{4} \right) \Big|_0^1 d\theta$$

$$= \frac{17\pi}{2} \quad \#$$

A final remark:

The derivation of formula (★) can be skipped.